

ON THE NORMALIZING ALGEBRA OF A MASA IN A II_1 FACTOR

IONUT CHIFAN

ABSTRACT. Let A be a maximal abelian subalgebra (MASA) in a II_1 factor M . Sorin Popa introduced an analytic condition that can be used to identify the normalizing algebra of A in M and which we call *the relative weak asymptotic homomorphism property*. In this paper we show this property is always satisfied by the normalizing algebra of A in M and as a consequence we obtain that $\overline{\bigotimes_{i \in I} (\mathcal{N}_{M_i}(A_i))''} = (\mathcal{N}_{\overline{\bigotimes_{i \in I} M_i}}(\overline{\bigotimes_{i \in I} A_i}))''$.

1. INTRODUCTION

Throughout this paper M will denote a fixed type II_1 factor and A will denote a maximal abelian subalgebra (MASA) of M . By $\mathcal{U}(M)$ we denote the unitary group of M and by $\mathcal{P}(M)$ the set of projections of M . Also, $\mathcal{N}_M(A)$ will denote the group of unitaries u in M such that $uAu^* = A$, i.e., $\mathcal{N}_M(A)$ is the *normalizer* of A in M . The von Neumann algebra $\mathcal{N}_M(A)''$ generated by $\mathcal{N}_M(A)$ will be called *the normalizing algebra of A in M* . We thus have

$$A \subseteq \mathcal{N}_M(A)'' \subseteq M.$$

A consistent study of algebra $\mathcal{N}_M(A)''$ started with the work of Jacques Dixmier. In [2] the author distinguished three cases of particular interest, according to the "size" of $\mathcal{N}_M(A)''$ in M :

- (1) *Regular (or Cartan)* if $M = \mathcal{N}_M(A)''$.
- (2) *Semi-regular* if $\mathcal{N}_M(A)''$ is a II_1 factor.
- (3) *Singular* if $A = \mathcal{N}_M(A)''$.

Our primary objective in this paper is to show in Theorem 2.4 that the triple

$$A \subseteq \mathcal{N}_M(A)'' \subseteq M$$

has the following property:

Definition 1.1. A triple of von Neumann algebras $B \subseteq N \subseteq M$, where M is a II_1 factor, is said to have *the relative weak asymptotic homomorphism property (relative WAHP)* when for all $x_1, x_2, x_3, \dots, x_n \in M$ and for every $\varepsilon > 0$ there exists $u \in \mathcal{U}(B)$ such that

$$\|E_B(x_i u x_j) - E_B(E_N(x_i) u E_N(x_j))\|_2 < \varepsilon$$

for $i, j = 1, \dots, n$. By E_B and E_N we denote the conditional expectations of M onto B and N , and by $\|x\|_2$ we denote the norm of x taken in $L^2(M)$ - the Hilbert space of the GNS construction based on the faithful normal trace of the factor M .

In [20] the authors introduced the notion of *weak asymptotic homomorphism property* (WAHP) referring to a MASA, $A \subset M$. We can immediately see this is equivalent to saying that the triple $A \subseteq A \subseteq M$ has the *relative WAHP*, so our Definition 1.1 can be viewed as a generalization of their concept.

Remarkably, in [19] it was shown that every singular MASA $A \subset M$ in a II_1 factor satisfies the WAHP, so our Theorem 2.4 is also a generalization of their result.

We would like to point out that Theorem 2.4 represents a suitable tool to compute the normalizing algebra of a MASA in certain situations of tensor products of II_1 factors:

Corollary 1.2. *Let I be a countable set, $\{M_i\}_{i \in I}$ a collection of II_1 factors and $\{A_i\}_{i \in I}$ a collection of abelian diffuse von Neumann algebras, such that for every $i \in I$ we have $A_i \subset M_i$ is a MASA. Then,*

$$\overline{\bigotimes_{i \in I} (\mathcal{N}_{M_i}(A_i))''} = (\mathcal{N}_{\overline{\bigotimes_{i \in I} M_i}}(\overline{\bigotimes_{i \in I} A_i}))''.$$

The same method can be used to estimate the normalizing algebra of certain subalgebras in cross-products (see Theorem 3.10).

In order to study singular MASAs, Sinclair and Smith [20] isolated a concept which they called *strong singularity*:

Definition 1.3. A MASA A in a type II_1 factor M is called *strongly singular* if for every $u \in \mathcal{U}(M)$,

$$\|E_{uAu^*} - E_A\|_{\infty, 2} \geq \|u - E_A(u)\|_2.$$

By a very beautiful argument, Lemma 2.1 from [18] shows that the WAHP of $A \subset M$ a MASA implies the strong singularity of A in M . Since we have in hand Theorem 2.4, the same argument shows the following:

Theorem 1.4. *(A generalization of Sinclair-Smith inequality) Let M be a II_1 factor with $A \subset M$ a MASA. Then,*

$$\|E_{uAu^*} - E_A\|_{\infty, 2} \geq \|u - E_{\mathcal{N}_M(A)}(u)\|_2, \text{ for all } u \in \mathcal{U}(M).$$

Here we would like to mention that even if the previous theorem is a generalization of strong singularity concept for a MASA in a II_1 factor, a more general version (with absolute

constant 1) for Theorem 6.2 in [17] remains open.

Even though the computations of the normalizing algebra of an arbitrary fixed subalgebra in a II_1 factor have proved to be a very difficult problem, in the last years we witnessed a constant and successful effort in this direction. Without making a formal definition, Popa ([15, 14, 16]) and independently Robertson-Sinclair-Smith ([20, 18]) verified WAHP for certain inclusions of von Neumann algebras $P \subset M$ and obtained containment of the normalizing algebra of P in M in various contexts. The case of free products and the case of weakly mixing actions of groups on von Neumann algebras are only few examples in this sense.

Furthermore, the ground breaking technology that Popa developed to control the normalizing algebra (and relative commutants in particular) works in a much more general setting ([15, 14]), involving intertwining elements between two distinct subalgebras rather than normalizing elements of a common subalgebra. More precisely, Theorem 2.1 and Corollary 2.3 in [14] give a complete description of the existence of intertwining elements between two distinct subalgebras in a fixed factor with discrete decomposition. This was called *intertwining techniques* and was one of the major ingredients that has led to many striking results in von Neumann Algebras theory/Noncommutative Ergodic theory([15, 14, 16, 6]).

However, in the situation $A \subset M$ is a MASA our Theorem 2.4 shows that relative WAHP intrinsically characterizes the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$. Our proof that the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$ satisfies the relative WAHP is based on a deep idea of S. Popa [15, 14], which is to build normalizing elements by looking at the relative commutant between the MASA and the basic construction [1, 7, 13] for the inclusion $A \subseteq M$. This connection has been made before with great benefit, but we refine it. Namely, we analyze the relationship between finite trace projections in the the basic construction $\langle M, e_A \rangle$ and the Jones projection e_N of the normalizing algebra N . It turns out that they satisfy an interesting geometric relation that is revealed in the proof of Proposition 2.6. This is the key observation of our proof.

This paper is organized in two sections. In the first section we prove Theorem 2.4, which is the main result of the paper. In the second section we present some immediate applications of this theorem described above (Proposition 3.5, Proposition 3.7, Corollary 3.8, Theorem 3.10).

Acknowledgments:

I would like to express my gratitude to my advisers, Paul Muhly and Florin Radulescu, for constant guidance and illuminating discussions about this material. I would also want to thank Fred Goodman for his enthusiastic encouragement during the last years. Last but

not the least I want to thank the referee and Prof. Roger Smith for the generous comments regarding the original version of this paper.

2. AN ANALYTIC CHARACTERIZATION OF NORMALIZING ALGEBRA OF A MASA IN A II_1 FACTOR

First we recall some elementary properties of the basic construction for an inclusion of two von Neumann algebras. These were thoroughly developed in [7, 1, 13, 11]. Then in Theorem 2.4 (which is the core result of this section) we provide an alternative description of the absence of normalizing elements for a MASA A in a II_1 factor M which we call *the relative weak asymptotic homomorphism property (WAHP)*.

Consider a triple $A \subseteq N \subseteq M$, where M is II_1 factor with the normalized trace τ , A a MASA, and N is an intermediate von Neumann subalgebra. The trace τ induces an inner product on M :

$$\langle x, y \rangle = \tau(y^*x) \quad \text{for all } x, y \in M$$

We denote by $L^2(M)$ the completion of M with respect to the norm $\|x\|_2 = (\tau(x^*x))^{1/2}$, and when an element $x \in M$ is seen as a vector in $L^2(M)$ it will be denoted by \hat{x} . Next, consider the τ -preserving conditional expectations E_A and E_N onto A and N , respectively. Each such conditional expectation can also be viewed as a projection in $B(L^2(M))$. Thus we can define:

$$e_A(\hat{x}) = \widehat{E_A(x)}$$

for all $x \in M$.

Also there is an anticonjugation $\mathcal{J} : L^2(M) \rightarrow L^2(M)$, defined by

$$\mathcal{J}(\hat{x}) = \widehat{x^*}$$

for $x \in M$.

Perform the basic construction with respect to A :

$$A \subset M \subset \langle M, e_A \rangle \subset B(L^2(M))$$

where $\langle M, e_A \rangle$ is the von Neumann algebra generated by M and e_A in $B(L^2(M))$, i.e. $\langle M, e_A \rangle = \{M, e_A\}''$.

In the next proposition we recall several important properties of this construction that are of essential use to our next proofs. For the reader convenience we also include references that provide detailed proofs and a complete history of these facts.

Proposition 2.1. *Suppose $A \subseteq N \subseteq M$ is a triple of algebras as before. Then the following properties hold true:*

- a. [7] We have $\langle M, e_A \rangle = \mathcal{JA}'\mathcal{J}$, which in particular says that $\langle M, e_A \rangle$ is a von Neumann algebra of type I_∞ .
- b. [7] For all $x \in M$ we have $e_A x e_A = E_A(x) e_A$. Moreover, $\overline{\text{span}}^w(M e_A M) = \langle M, e_A \rangle$, and if $N = \mathcal{N}_M(A)$, then $e_N \in A \vee \mathcal{JA}\mathcal{J}$. (by $A \vee \mathcal{JA}\mathcal{J}$ we denote the von Neumann algebra generated by A and $\mathcal{JA}\mathcal{J}$ and by $\overline{\text{span}}^w(M e_A M)$ the w -closure of the linear span of $M e_A M$)
- c. [13, 11] There exists a semifinite trace Tr on $\langle M, e_A \rangle$ determined by the equation $\text{Tr}(x e_A y) = \tau(xy)$ for all $x, y \in M$.
- d. [13, 11] (Pull down identity) We have $e_A \langle M, e_A \rangle = \overline{e_A M}^w$ and $\langle M, e_A \rangle e_A = \overline{M e_A}^w$. More precisely, there exists an M - M bimodule map $\Phi : L^1(\langle M, e_A \rangle, \text{Tr}) \rightarrow L^1(M, \tau)$ which satisfies the pull down identity: if $V \in \langle M, e_A \rangle$, then $e_A V = e_A \Phi(e_A V)$. Moreover, $\Phi(\langle M, e_A \rangle) \subseteq L^2(M)$.

Next, we introduce the following definition:

Definition 2.2. A triple of von Neumann algebras $B \subseteq N \subseteq M$, where M is a II_1 factor, has the relative weak asymptotic homomorphism property (relative WAHP) when for all $x_1, x_2, x_3, \dots, x_n \in M$, and for every $\varepsilon > 0$ there exists $u \in \mathcal{U}(B)$ such that:

$$\|E_B(x_i u x_j) - E_B(E_N(x_i) u E_N(x_j))\|_2 < \varepsilon$$

for $i, j = 1, \dots, n$.

For brevity, we will often refer to the relative weak asymptotic homomorphism property as relative WAHP. We were motivated to call this property *relative weak asymptotic homomorphism property* because it is a relative version for Robertson-Sinclair-Smith's notion of *weak asymptotic homomorphism property* of a singular MASA. These facts will be more amply discussed in section 3. In connection to the Definition above we record the following proposition:

Proposition 2.3. Let M be II_1 factor with $A \subseteq N \subseteq M$ two von Neumann subalgebras. The triple $A \subseteq N \subseteq M$ satisfy the relative weak asymptotic homomorphism property if and only if it satisfy the following property:

For all $x_1, x_2, \dots, x_n \in M$, with $E_N(x_i) = 0$ and for every $\varepsilon > 0$ there exists $u \in \mathcal{U}(A)$ such that

$$\|E_A(x_i u x_j)\|_2 < \varepsilon$$

for all $i, j = 1, 2, \dots, n$

Proof. The equivalence between relative weak asymptotic homomorphism property follows immediately if we use the identity:

$$E_A(xuy) - E_A(E_N(x)uE_N(y)) = E_A((x - E_N(x))u(y - E_N(y)))$$

□

Next, we state the main theorem of this section:

Theorem 2.4. *Let M be II_1 factor and let $A \subset M$ be a MASA. Then the triple*

$$A \subseteq \mathcal{N}_M(A)'' \subseteq M$$

satisfies the relative weak asymptotic homomorphism property.

The proof of this theorem will be a consequence of the next sequence of lemmas and propositions.

Lemma 2.5. *Let M be II_1 factor and let $A \subset M$ be a MASA. Denote by $N := \mathcal{N}_M(A)''$ and assume that the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$ does not satisfy relative WAHP. Then, there exists a non-zero projection $f \in \mathcal{P}(A' \cap \langle M, e_A \rangle)$ such that $fe_N = 0$, and $\text{Tr}(f) < \infty$.*

Proof. The proof we present is essentially the proof of Corollary 2.3 in [15] with very slight changes, and we reproduced it here for the sake of completeness.

If the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$ does not satisfy relative WAHP then, by Proposition 2.2, there exist $x_1, x_2, \dots, x_n \in M$ such that: $E_N(x_i) = 0$ for $i = 1, 2, \dots, n$ and there exists $\varepsilon_0 > 0$ such that

$$\sum_{i,j=1}^n \|E_A(x_i u x_j^*)\|_2^2 \geq \varepsilon_0$$

for all $u \in \mathcal{U}(A)$. Using Proposition 2.1 c., a little computation shows that the last inequality is equivalent to $(\alpha) \quad \text{Tr}(bubu^*) \geq \varepsilon_0$ for all $u \in \mathcal{U}(A)$, where $b = \sum_{i=1}^n x_i^* e_A x_i$.

Denote by $K(b) = \overline{\text{co}}^w \{ubu^* | u \in \mathcal{U}(A)\}$ a weak-operator-compact convex set in $\langle M, e_A \rangle$. If a is the unique $\|\cdot\|_{2,\text{Tr}}$ -minimal element in $K(b)$ then:

$$0 \leq a \leq 1, \quad \text{Tr}(a) \leq \text{Tr}(b) \leq \infty, \quad \text{and } a \in A' \cap \langle M, e_A \rangle.$$

Also (α) obviously implies that $\text{Tr}(bx) \geq \varepsilon_0$ for all $x \in K(b)$. In particular $\text{Tr}(ba) \geq \varepsilon_0$ implies $a \neq 0$. Since we are assuming that $E_N(x_i) = 0$ for all $i = 1, 2, \dots, n$, there are few more conditions, besides the details from the proof of Corollary 2.3 in [15] that we need to check before we are able to derive our conclusion. Namely:

$$be_N = \sum_{i=1}^n x_i^* e_A x_i e_N = \sum_{i=1}^n x_i^* e_A e_N x_i e_N = \sum_{i=1}^n x_i^* e_A E_N(x_i) = 0,$$

and because $e_N \in N' \subset A'$, we have

$$\begin{aligned}
ubu^*e_N &= 0 \\
\Rightarrow \text{co}(ubu^*)e_N &= 0 \\
\Rightarrow K(b)e_N &= 0 \quad \text{and likewise } e_N K(b) = 0. \\
\Rightarrow e_N a &= a e_N = 0 \text{ and } 0 \leq a \leq 1 \\
\Rightarrow 0 &\leq a \leq 1 - e_N.
\end{aligned}$$

Finally, by taking a suitable spectral projection of a in the algebra $(1 - e_N)A' \cap \langle M, e_A \rangle (1 - e_N)$ we find a nonzero projection $f \in P(A' \cap \langle M, e_A \rangle)$ such that $f e_N = 0$, and $\text{Tr}(f) < \infty$. \square

Lemma 2.6. *Let M be II_1 factor and let $A \subset M$ be a MASA. Suppose there exists $f \in A' \cap \langle M, e_A \rangle$, a nonzero projection that satisfies $\text{Tr}(f) < \infty$. Then there exist nonzero projections $p_i \in A' \cap \langle M, e_A \rangle$ that are abelian in $\langle M, e_A \rangle$ for all $i \in I$ and which further satisfy the following equation:*

$$f = \sum_{i \in I} p_i$$

Proof. Denote by $\mathcal{A} := A \vee \mathcal{J}A\mathcal{J}$ and consider the inclusions:

$$\begin{aligned}
\mathcal{A}' &= A' \cap \langle M, e_A \rangle \subseteq \langle M, e_A \rangle = (\mathcal{J}A\mathcal{J})' \\
fA' \cap \langle M, e_A \rangle f &\subset f\langle M, e_A \rangle f \\
\mathcal{Z}(\langle M, e_A \rangle) &= \mathcal{J}A\mathcal{J}
\end{aligned}$$

First note that both algebras $A' \cap \langle M, e_A \rangle$ and $\langle M, e_A \rangle$ are of type I_∞ . Since $f \in \mathcal{A}'$ is a finite projection, both algebras $fA' \cap \langle M, e_A \rangle f$ and $f\langle M, e_A \rangle f$ are finite of type I . Also note that the central support of e_A in $\langle M, e_A \rangle$ is equal to 1. ($z_{\langle M, e_A \rangle}(e_A) = 1$)

By general theory, we have $f\langle M, e_A \rangle f \cong \sum_{j \in J} A_j \bar{\otimes} M_{n_j}(\mathbb{C})$. This implies that $\mathcal{A}f$ is an abelian subalgebra of $\sum_{j \in J} A_j \otimes M_{n_j}(\mathbb{C})$.

Consequently, $\mathcal{A}f z_j \subset A_j \bar{\otimes} M_{n_j}(\mathbb{C})$ (where $z_j \in \mathcal{J}A\mathcal{J}$) and there exists a MASA B_j , such that $f\mathcal{A}z_j \subset B_j \subset A_j \bar{\otimes} M_{n_j}(\mathbb{C})$.

By Kadison's result [8], we have $uB_j u^* = A_j \otimes D_{n_j}(\mathbb{C})$, where $u \in \mathcal{U}(A_j \bar{\otimes} M_{n_j}(\mathbb{C}))$. That implies $fz_j = \sum_{i=1}^{n_j} p_i^j$ where $p_i^j \in (f\mathcal{A}z_j)' \cap fz_j A_j \otimes M_{n_j}(\mathbb{C}) z_j f$. But this shows that p_i^j actually belongs to $\mathcal{A}' f z_j$ and, moreover, is abelian in $\langle M, e_A \rangle$ (i.e., $p_i^j \langle M, e_A \rangle p_i^j = z_j p_i^j f \langle M, e_A \rangle f p_i^j z_j = A_j \otimes \mathbb{C} p_i^j$ abelian algebra).

But this implies $f = \sum f z_j = \sum_{j \in \overline{1, n_j}} p_i^j$ which completes the proof of this step. In particular, we have: if f is a nonzero projection in $A' \cap \langle M, e_A \rangle$ with $\text{Tr}(f) < \infty$, then

there exists a nonzero projection p in $A' \cap \langle M, e_A \rangle$ which is abelian in $\langle M, e_A \rangle$ and satisfies $p \leq f$. \square

Proposition 2.7. *Let A be a masa in the II_1 factor M and let $N := \mathcal{N}_M(A)''$. If p is a non-zero projection in $A' \cap \langle M, e_A \rangle$ with $p \preceq e_A$ as projections in $\langle M, e_A \rangle$, then exists a non-zero projection q with $q \leq p$ and $q \leq e_N$.*

Proof. Since $p \preceq e_A$, let $W \in \langle M, e_A \rangle$ be a partial isometry such that $p = W^*W$, $WW^* \leq e_A$, $Wp = W$, $e_AW = W$.

The property $e_A \langle M, e_A \rangle e_A = Ae_A$ (Proposition 2.1 c) implies there exists an well defined function $\phi : A \rightarrow A$ given by the equation:

$$e_A W a W^* e_A = \phi(a) e_A.$$

We remark that ϕ is a $*$ -homomorphism. Indeed, it satisfies the following:

- $e_A W(a+b)W^* e_A = \phi(a+b)e_A$ and $e_A W a W^* e_A + e_A W b W^* e_A = (\phi(a) + \phi(b))e_A \Rightarrow \phi(a+b) = \phi(a) + \phi(b)$ by Proposition 2.1 c.
- $\phi(a^*)e_A = e_A W a^* W^* e_A = (e_A W a W^* e_A)^* = \phi(a)^* e_A \Rightarrow \phi(a^*) = \phi(a)^*$.
- $\phi(ab)e_A = e_A W ab W^* e_A = e_A W W^* e_A W ab W e_A = e_A W a W^* e_A W b W e_A = \phi(a)e_A \phi(b)e_A = \phi(a)\phi(b)e_A \Rightarrow \phi(ab) = \phi(a)\phi(b)$ by Proposition 2.1 c. again. Notice we also used here that $W^* e_A W \in A'$.

Since $p = W^* e_A W \in A' \cap \langle M, e_A \rangle$ we have $ap = pa$ for all $a \in A$. This implies $W^* e_A W a = a W^* e_A W$ and because $e_A W = W$ is an isometry we obtain $W a = e_A W a W^* e_A W$, which we rewrite as

$$W a = \phi(a) W$$

for all $a \in A$.

For a more detailed account on $*$ -homomorphism ϕ we send the reader to [15, 17].

Since $\text{span} M e_A M$ is weakly dense $*$ -algebra in $\langle M, e_A \rangle$, using Kaplansky density theorem, there exists $(z_n)_n \in \text{span} M e_A M$ such that $z_n \rightarrow W^*$ in so -topology. But $W^* = W^* e_A$ implies that $z_n e_A \rightarrow W^* e_A = W^*$. Using $e_A M e_A = A e_A$ we see that $z_n e_A = y_n e_A$ with $y_n \in M$. Next, denote by $\eta = \mathcal{J} W^* \hat{1} \in L^2(M, \tau)$ where $\hat{1}$ is the canonical cyclic trace vector for the left regular representation associated with τ .

Also by l_η we mean the left multiplication operator by η defined on $M \hat{1}$. It is well known this operator is closable and we denote by $L_\eta := \overline{l_\eta}$ its closure. We record that L_η is a closed densely defined operator affiliated with M .

Next equation establishes the relation between W and L_η , which is known in the literature as the pull-down identity (see [13, 11]):

$$e_A W x \hat{1} = e_A L_\eta(x \hat{1}) \quad (2.6.1.)$$

for all $x \in M$.

To verify this we observe that it is enough to check the following:

$$\begin{aligned} \langle e_A L_\eta(x \hat{1}), y \hat{1} \rangle &= \langle L_\eta(x \hat{1}), E_A(y) \hat{1} \rangle \\ &= \langle x \hat{1}, L_{\mathcal{J}\eta}(E_A(y) \hat{1}) \rangle = \langle x \hat{1}, \mathcal{J} E_A(y^*) \mathcal{J} W^* \hat{1} \rangle \\ &= \lim_n \langle x \hat{1}, \mathcal{J} E_A(y^*) \mathcal{J} z_n \hat{1} \rangle = \lim_n \langle x \hat{1}, z_n E_A(y) \hat{1} \rangle \\ &= \lim_n \langle x \hat{1}, z_n e_A E_A(y) \hat{1} \rangle = \lim_n \langle x \hat{1}, z_n e_A E_A(y) \hat{1} \rangle \\ &= \langle x \hat{1}, W^* E_A(y) \hat{1} \rangle = \langle W x \hat{1}, E_A(y) \hat{1} \rangle = \langle e_A W x \hat{1}, y \hat{1} \rangle \text{ for every } x, y \in M. \end{aligned}$$

From now on, whenever two unbounded operators S and T defined on $L^2(M, \tau)$ agree on $M \hat{1}$, we write $S \doteq T$. Consequently, we can rewrite the equation (2.6.1.) as:

$$e_A W \doteq e_A L_\eta.$$

Further, we can also check the following:

$$\begin{aligned} L_\eta a(x \hat{1}) &= L_\eta((ax) \hat{1}) \\ &= \mathcal{J}(ax)^* \mathcal{J} \eta = \mathcal{J} x^* a^* \mathcal{J} \mathcal{J} W^* \hat{1} \\ &= \mathcal{J} x^* a^* W^* \hat{1} = \mathcal{J} x^* W^* \phi(a^*) \hat{1} \\ &= \mathcal{J} x^* W^* \mathcal{J} \phi(a) \hat{1} = \phi(a) \mathcal{J} x^* \mathcal{J} \mathcal{J} W^* \mathcal{J} \hat{1} \\ &= \phi(a) \mathcal{J} x^* \mathcal{J} \eta = \phi(a) L_\eta(x \hat{1}) \text{ for every } a \in A, x \in M. \end{aligned}$$

In other words we have proved that:

$$L_\eta a \doteq \phi(a) L_\eta \quad (2.6.2.)$$

for all $a \in A$.

Let $L_\eta = wT$ be the polar decomposition of L_η where $w \in M$ is the partial isometry mapping the closure of the range of T to the closure of the range of L_η and $T = |L_\eta|$ is the absolute value of L_η .

The relation (2.6.2.) becomes

$$wTa \doteq \phi(a)wT \quad (2.6.3.)'$$

which by the same argument like in the Lemma 5.1 from [17] implies that

$$w^*wa = w^*\phi(a)w \quad (2.6.3.)$$

for all $a \in A$.

In particular this equation carries the fact that $w^*w \in A$ and, moreover, it can be proved that $ww^* \in \phi(A)' \cap M$.

To see this let $f \in \mathcal{P}(A)$ be an arbitrary fixed projection. Then, $w^*\phi(f)ww^*\phi(f)w = w^*wfw^*wf = w^*wf^2 = w^*wf = w^*\phi(f)w$, which further implies that $ww^*\phi(f)(1 - ww^*) = 0$ and consequently $ww^*\phi(f) = \phi(f)ww^*$. Since this last equation holds true for any projection $f \in A$ our conclusion follows. So the equation (2.6.3.) is actually equivalent to

$$wa = \phi(a)w \quad (2.6.4.)$$

for all $a \in A$.

Also, if we further combine (2.6.4.) with the fact that $w^*wT \dot{=} T$ then the equation (2.6.3.)' implies:

$$aT \dot{=} Ta \quad (2.6.5)$$

for all $a \in A$.

Now consider $\mathcal{D}_1 := \mathcal{D}(\overline{aT}) \cap \mathcal{D}(\overline{Ta}) \cap \mathcal{D}(T)$, where by $\mathcal{D}(\overline{aT})$ we denoted the domain of the (closed) operator \overline{aT} and so on....

Because $M\hat{1} \subset \mathcal{D}(T) \subset \mathcal{D}(aT) \subset \mathcal{D}(\overline{aT})$ and $M\hat{1} \subset \mathcal{D}(Ta) \subset \mathcal{D}(\overline{Ta})$, we have that $M\hat{1} \subset \mathcal{D}_1$. Now by Lemma 16.4.3 in [9] we have that \mathcal{D}_1 is essentially dense and everywhere dense in $L^2(M, \tau)$ and moreover we prove below that the equality (2.6.5.) actually holds true on \mathcal{D}_1 .

If $\zeta \in \mathcal{D}_1$ then there exists $m_n\hat{1} \in M\hat{1}$ such that $m_n\hat{1} \rightarrow \zeta$. Next, the equation (2.6.5.) implies that $\langle m_n\hat{1}, \overline{aT} - \overline{Ta}(m_n\hat{1}) \rangle = \langle m_n\hat{1}, aT - Ta(m_n\hat{1}) \rangle = \langle m_n\hat{1}, 0 \rangle \rightarrow \langle \zeta, 0 \rangle$ which further gives that $\overline{aT} - \overline{Ta}(\zeta) = 0$ and so $\overline{aT}(\zeta) = \overline{Ta}(\zeta)$ for every $\zeta \in \mathcal{D}_1$. Since $Ta = \overline{Ta}$ and $\mathcal{D}(T) \subset \mathcal{D}(aT)$ we conclude that

$$aT(\zeta) \dot{=} Ta(\zeta) \quad (2.6.6.)$$

for all $\zeta \in \mathcal{D}_1$ and all $a \in A$.

Now consider $\mathcal{D}_2 := \mathcal{D}_1 \cap \mathcal{D}(\overline{T^2}) \subset \mathcal{D}_1$. By Lemma 16.4.3 [9] again we have that \mathcal{D}_2 is essentially dense and everywhere dense in $L^2(M, \tau)$.

Next we verify that $T^2 + a^2|_{\mathcal{D}_2}$ is essentially selfadjoint. On one hand we have $\langle T^2 + a^2\zeta, \psi \rangle = \langle \zeta, \overline{T^2} + a^2\psi \rangle$ and hence

$(T^2 + a^2|_{\mathcal{D}_2})^* \subset \overline{T^2 + a^2|_{\mathcal{D}(\overline{T^2})}}$ which is obviously a closed operator. Using the uniqueness of the extension from Lemma 16.4.2 [9] we see that

$(T^2 + a^2|_{\mathcal{D}_2})^* = \overline{T^2 + a^2|_{\mathcal{D}(\overline{T^2})}}$. On the other hand, $\overline{T^2 + a^2|_{\mathcal{D}_2}} \subset \overline{\overline{T^2} + a^2|_{\mathcal{D}(\overline{T^2})}} = \overline{T^2} + a^2|_{\mathcal{D}(\overline{T^2})}$ and by uniqueness again we have $\overline{T^2 + a^2|_{\mathcal{D}_2}} = \overline{T^2} + a^2|_{\mathcal{D}(\overline{T^2})}$.

In conclusion $T^2 + a^2|_{\mathcal{D}_2}$ is essentially selfadjoint. Combining this with the fact that T and a commutes on \mathcal{D}_2 (see (2.6.6.)), by Corollary 9.2 in [10] we obtain that T and a

strongly commute which means their spectral scales commutes. So the spectral scale of T belongs to $A' \cap M = A$.

To this end we prove the following:

*Claim: There exists $m \in M$ such that m^*e_Am is a nonzero projection in $A' \cap \langle M, e_A \rangle$ that satisfies $m^*e_Am \leq p$.*

To show this let f be a spectral projection of T such that $0 \neq fT = Tf \in M$. By (2.6.1.) we have $Wf(x\hat{1}) = e_AW(fx\hat{1}) = e_AL_\eta(fx\hat{1}) = e_AwT(fx\hat{1}) = e_Aw(Tf)(x\hat{1})$. Since Tf is a bounded operator that belongs to M we have obtained that $Wf = e_Am$ for some $m \in M$.

Also by the choice of f we have that $0 \neq Wf$ so we can verify the following: $0 \neq m^*e_Am = fW^*Wf = W^*Wf^2 = W^*WfW^*W \leq (W^*W)^2 = p$

$$m^*e_Am = fW^*Wf = (W^*Wf)^2 = (me_Am^*)^2.$$

$$m^*e_Am = fW^*Wf = W^*Wf \in A' \cap \langle M, e_A \rangle.$$

This finishes the proof of the *Claim*.

Since $m^*e_Am \in A'$ we have $m^*e_Ama = am^*e_Am$ for all $a \in A$, which is equivalent to

$$ma = E_A(mam^*)m \quad (2.6.7.)$$

for all $a \in A$.

Because relation (2.6.7.) holds for every $a \in A$ by considering the "stared" version of it we obtain $m^*m \in A' \cap M = A$ and hence $|m| \in A$.

Next, denote by $h = \chi_{(\infty, 1)}(|m|) \in A$ the spectral projection of the element $|m|$ corresponding to the interval $(\infty, 1)$. To this end we split the proof of this proposition in two cases:

CASE I $h \neq 0$.

Relation (2.6.7.) implies that $mha = E_A(mhahm^*)mh$ (2.6.8) for all $a \in A$. Consider $m_1 = mh$ and using the spectral properties of h we see $m_1^*m_1 = hm^*mh \leq h \leq 1$ which is equivalent to $m_1m_1^* \leq 1$. Also, by plugging in $a = 1$ in equation (2.6.8.) we have $m_1 = E_A(m_1m_1^*)m_1$ which in particular implies that $E_A(m_1m_1^*) \in \mathcal{P}(A)$.

Next, notice $m_1m_1^* = E_A(m_1m_1^*)m_1m_1^*E_A(m_1m_1^*) \leq (E_A(m_1m_1^*))^2 = E_A(m_1m_1^*)$ and by the τ -invariance of E_A and the faithfulness of τ we conclude $m_1m_1^* = E_A(m_1m_1^*)$ (2.6.9.).

Also we remark that the equation (2.6.8.) implies that $m_1^*m_1 \in A$ and this together with equations (2.6.9.) help us to conclude that $m_1 \in \mathcal{GN}_M(A)$. Using the structure of $\mathcal{GN}_M(A)$ ([3]) there exists $u \in \mathcal{N}_M(A)$, $e \in A$ such that $m_1 = ue$ and we can verify that $m_1^*e_Am_1 = P_{m_1^*A} = P_{eu^*A} \leq P_N = e_N$.

But $h \neq 0$ implies $m_1^* e_A m_1 \neq 0$ and also we can check that $m_1^* e_A m_1 = h m^* e_A m h = m^* e_A m h m^* e_A m \leq m^* e_A m \leq p$. Hence in this case the proof of the proposition is finished.

CASE II $h = 0$.

If $h = 0$, then $\sigma(|m|) \subset [1, \infty)$ so in particular $|m|$ is invertible and $1 \leq m^* m$ (2.6.10.). By taking the polar decomposition of $m = v|m|$ we have that $v^* v = \text{supp}(|m|) = 1$ which implies $v \in \mathcal{U}(M)$ because M is a finite factor.

Moreover, the relation (2.6.7.) becomes $v|m|a = E_A(mam^*)v|m|$ and so $va|m| = E_A(mam)^*v|m|$ for all $a \in A$. By multiplying on the right by $|m|^{-1}$ we get $va = E_A(mam^*)v$ which we rewrite as $vav^* = E_A(mam^*)$ for all $a \in A$. This last equation implies $vAv^* \subseteq A$ which together with $v \in \mathcal{U}(M)$ and A is a masa in M further implies that $v \in \mathcal{N}_M(A)$.

From this we notice that $0 \neq v^* e_A v = P_{v^* A} \leq e_N$ (2.6.11.) and $v^* e_A v \in A' \cap \langle M, e_A \rangle$.

On the other hand using (2.6.10.) we can check the following

$$0 \neq v^* e_A v \leq v^* e_A v m^* m v^* e_A v = v^* e_A v |m|^2 v^* e_A v = |m| v^* e_A v |m| = m^* e_A m^* \leq p \quad (2.6.12.)$$

Relations (2.6.11) and (2.6.12.) finish the proof in this case. \square

We end this section by presenting *the proof of the Theorem 2.3*:

Proof. : We will proceed by contradiction. Let suppose that the triple does not satisfy the relative WAHP. By Lemma 2.4 there exists a nonzero projection $f \in \mathcal{P}(A' \cap \langle M, e_A \rangle)$ with $\text{Tr}(f) < \infty$ and $f e_N = 0$ (\dagger). Moreover, using Lemma 2.5 there exists a nonzero projection $p \leq f$ which lies in $A' \cap \langle M, e_A \rangle$ and is abelian $\langle M, e_A \rangle$. But we obviously have $z_{\langle M, e_A \rangle}(p) \leq 1 = z_{\langle M, e_A \rangle}(e_A)$ which further implies that $p \preceq e_A$. By Proposition 2.6 there exists a non-zero projection q such that $q \leq p \leq f$ and $q \leq e_N$, which is in contradiction with (\dagger). In conclusion the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$ must satisfy the relative WAHP. \square

3. APPLICATIONS

In this section we present several immediate applications of Theorem 2.4.

The first result of the section underlines the fact that the *relative weak asymptotic homomorphism property* (relative WAHP) for a triple of algebras is a suitable tool to control the normalizing algebra of a given subalgebra. This idea was exploited before in [18] (see Lemma 2.1), where the authors proved that the *weak asymptotic homomorphism property* (WAHP) of a MASA implies the strong singularity of that MASA. The same argument can be used to prove the following:

Proposition 3.1. *Let M be a II_1 factor and $A \subseteq N \subseteq M$ two von Neumann subalgebras. If the triple $A \subseteq N \subseteq M$ satisfies the relative weak asymptotic homomorphism property then the following inequality holds for every $u \in \mathcal{U}(M)$:*

$$\|E_{uAu^*} - E_A\|_{\infty,2} \geq \|u - E_N(u)\|_2$$

Proof. If we apply relative WAHP for the set $\{u, u^*\}$ and $\varepsilon > 0$ arbitrary but fixed, then there is $a_\varepsilon \in \mathcal{U}(A)$ such that:

$$(\gamma) \quad \|E_A(u^*a_\varepsilon u) - E_A(E_N(u^*)a_\varepsilon E_N(u))\|_2 < \varepsilon.$$

But,

$$\begin{aligned} \|E_{uAu^*} - E_A\|_{\infty,2}^2 &\geq \|E_{uAu^*}(a_\varepsilon) - E_A(a_\varepsilon)\|_2^2 \\ &= 1 - \|E_A(u^*a_\varepsilon u)\|_2^2 \text{ (because } a_\varepsilon \in \mathcal{U}(A)) \\ &\geq 1 - (\|E_A(E_N(u^*)a_\varepsilon E_N(u))\|_2 + \varepsilon)^2 \text{ (by } (\gamma)) \\ &\geq 1 - (\|E_N(u^*a_\varepsilon E_N(u))\|_2 + \varepsilon)^2 \text{ (since } e_A \text{ is a projection in } B(L^2(M))) \\ &\geq 1 - (\|E_N(u)\|_2 + \varepsilon)^2 \text{ (} E_A \text{ is a Schwartz map, } a_\varepsilon \in \mathcal{U}(A), u \in \mathcal{U}(M)) \\ &\geq \|u - E_N(u)\|_2^2 - \varepsilon(\varepsilon + 2) \end{aligned}$$

Note this is true for any $\varepsilon > 0$ so by taking $\varepsilon \rightarrow 0$ we obtain the desired result. \square

Corollary 3.2.

Let M be a II_1 factor with A and N two von Neumann subalgebras. If we assume that the triple $A \subseteq N \subseteq M$ satisfies the relative WAHP, then we have $\mathcal{N}_M(A)'' \subseteq N$.

Proof. The proof is an obvious consequence of the previous proposition. \square

Theorem 3.3.

(A generalization of Sinclair-Smith inequality) Let M be a II_1 factor with $A \subset M$ a MASA. Then,

$$\|E_{uAu^*} - E_A\|_{\infty,2} \geq \|u - E_{\mathcal{N}_M(A)}(u)\|_2, \text{ for all } u \in \mathcal{U}(M).$$

Proof. By Theorem 2.4 the triple $A \subseteq \mathcal{N}_M(A)'' \subseteq M$ satisfies the relative WAHP so the statement follows from Proposition 3.1. \square

At this point we would like to mention that even though the previous theorem is a generalization of strong singularity concept for a MASA in a II_1 factor, a more general version (with absolute constant 1) for Theorem 6.2 in [17] remains open.

In the last part of this section we will present a series of estimates of the normalizing algebra of a MASA in the situations of tensor products and cross products by discrete

groups. These estimates heavily rely on relative WHP for certain triples of algebras. Before starting we state an alternative description of relative WHP for a triple of algebras which is more convenient to use in our future computations.

Remark 3.4. Let M be a II_1 factor and $A \subseteq N \subseteq M$ two von Neumann subalgebras. Let $\mathcal{X} \subset M$ such that $\text{span}\mathcal{X}$ is a $*$ -subalgebra which is weakly dense in M . The triple $A \subseteq N \subseteq M$ satisfies the *relative weak asymptotic homomorphism property* if and only if:

For all $x_1, x_2, \dots, x_n \in \mathcal{X}$, and for every $\varepsilon > 0$ there exists $u \in \mathcal{U}(A)$ such that

$$\|E_A(x_i u x_j) - E_A(E_N(x_i) u E_N(x_j))\|_2 < \varepsilon$$

for $i, j = 1, \dots, n$.

Proposition 3.5. *Let M_1 and M_2 be II_1 factors. For $i = 1, 2$, let A_i be a MASA in M_i . Then,*

$$\mathcal{N}_{M_1 \overline{\otimes} M_2}(A_1 \overline{\otimes} A_2)'' = \mathcal{N}_{M_2}(A_2)'' \overline{\otimes} \mathcal{N}_{M_1}(A_1)''.$$

In particular if A_i is a singular MASA in M_i for $i = 1, 2$ then $A_1 \overline{\otimes} A_2 \subset M_1 \overline{\otimes} M_2$ is a singular MASA.

Proof. For each $i = 1, 2$, we denote by $N_i := \mathcal{N}_{M_i}(A_i)''$. We only need to prove

$$\mathcal{N}_{M_1 \overline{\otimes} M_2}(A_1 \overline{\otimes} A_2)'' \subseteq \mathcal{N}_{M_2}(A_2)'' \overline{\otimes} \mathcal{N}_{M_1}(A_1)'',$$

the other containment being trivial.

By Corollary 3.3 to show this would be enough to prove that the triple:

$$A_1 \overline{\otimes} A_2 \subseteq N_1 \overline{\otimes} N_2 \subseteq M_1 \overline{\otimes} M_2$$

satisfies the relative WHP.

Further, since $\mathcal{X} = \text{span}\{x \otimes y \mid x \in M_1, y \in M_2\}$ is $\|\cdot\|_2$ -dense in $M_1 \overline{\otimes} M_2$ the triple $A_1 \overline{\otimes} A_2 \subseteq N_1 \overline{\otimes} N_2 \subseteq M_1 \overline{\otimes} M_2$ satisfies the relative WHP iff we have the following:

(*) For all $x_1 \otimes y_1, \dots, x_n \otimes y_n \in \mathcal{X}$ and every $\varepsilon > 0$ exists $a \in \mathcal{U}(A_1 \overline{\otimes} A_2)$ such that:

$$\|E_{A_1 \overline{\otimes} A_2}((x_i \otimes y_i) a (x_j \otimes y_j)) - E_{A_1 \overline{\otimes} A_2}(x_i \otimes y_i a E_{N_1 \overline{\otimes} N_2}(x_j \otimes y_j))\|_2 < \varepsilon$$

for all $i, j = 1, \dots, n$.

By Theorem 2.4 we have that the triples $A_1 \subseteq N_1 \subseteq M_1$ and $A_2 \subseteq N_2 \subseteq M_2$ satisfies the relative WHP, so there exists $a_1 \in \mathcal{U}(A_1)$, $a_2 \in \mathcal{U}(A_2)$ which satisfies the following inequalities:

$$\|E_{A_1}(x_i a_1(x_j - E_{N_1}(x_j)))\|_2 < \frac{\varepsilon}{2 \max_{i=1, \dots, n} \|y_i\|^2}$$

for all $i, j = 1, \dots, n$ and

$$\|E_A(y_i a_2(y_j - E_{N_2}(y_j)))\|_2 < \frac{\varepsilon}{2 \max_{i=1, \dots, n} \|y_i\|^2}$$

for all $i, j = 1, \dots, n$.

Finally, we evaluate:

$$\begin{aligned} & \|E_{A_1 \overline{\otimes} A_2}((x_i \otimes y_i)(a_1 \otimes a_2)(x_j \otimes y_j - E_{N_1 \overline{\otimes} N_2}(x_j \otimes y_j)))\|_2 \\ &= \|E_{A_1}(x_i a_1(x_j - E_{N_1}(x_j))) \otimes E_{A_2}(y_i a_2 y_j) + E_{A_1}(x_i a_1 E_{N_1}(x_j)) \otimes E_{A_2}(y_i a_2(y_j - E_{N_2}(y_j)))\|_2 \\ &\leq \max_{i=1, \dots, n} \|y_i\|^2 \|E_{A_1}(x_i a_1(x_j - E_{N_1}(x_j)))\|_2 + \max_{i=1, \dots, n} \|x_i\|^2 \|E_{A_2}(y_i a_2(y_j - E_{N_2}(y_j)))\|_2 \\ &< \varepsilon \end{aligned}$$

This completes the proof of (*) and Proposition 3.5. □

Corollary 3.6. *If M_i is a II_1 factor and $A_i \subseteq M_i$ is a MASA for every $i = 1, \dots, k$ then we have:*

$$\overline{\otimes}_{i=1}^k \mathcal{N}_{M_i}(A_i)'' = \mathcal{N}_{\overline{\otimes}_{i=1}^k M_i}(\overline{\otimes}_{i=1}^k A_i)'' \quad (**)$$

Proof. The proof follows from Proposition 3.5 by induction. □

At this point it is natural to investigate if (**) holds true for infinite tensor products. As expected, the answer is yes but to be able to prove this we first need to analyze the behavior of the normalizing algebra with respect to the inductive limit.

Proposition 3.7. *Let M_n , $n \in \mathbb{N}$, be an increasing sequence of II_1 factors that are contained in a larger II_1 factor Q . Let $A_n \subseteq M_n$ be a MASA, and suppose that $A_n \subseteq A_{n+1}$ and moreover*

$$\begin{array}{ccc} M_{n+1} & \supset & A_{n+1} \\ \cup & & \cup \\ M_n & \supset & A_n \end{array}$$

is a commuting square for all $n \in \mathbb{N}$. Denote by $P_n = \mathcal{N}_{M_n}(A_n)''$, by $M = \overline{\cup_n M_n}^w$ and by $A = \overline{\cup_n A_n}^w$. Then $A \subset N$ is MASA and $\mathcal{N}_M(A)'' \subseteq \bigvee_n P_n$.

Proof. : It is helpful to keep the following diagram in mind:

$$\begin{array}{ccccc}
M & \supset & \bigvee_n P_n & \supset & A \\
\cup & & & & \cup \\
\vdots & & & & \vdots \\
\cup & & & & \cup \\
M_{n+1} & \supset & \bigvee_{s=1}^{n+1} P_s & \supset & A_{n+1} \\
\cup & & & & \cup \\
M_n & \supset & \bigvee_{s=1}^n P_s & \supset & A_n
\end{array}$$

We will only prove the second statement, the first one being nothing but Proposition 5.2.2 in [12]. Write $P = \bigvee_n P_n$. In order to prove that $\mathcal{N}_M(A)'' \subseteq P$, it is enough to show that the triple $A \subseteq P \subseteq M$ satisfies the relative WAHP, which by Remark 3.4 reduces to showing the following:

For every $\varepsilon > 0$, for each $n \in N$ and for all $x_1, \dots, x_k \in M_n$ there exists $a \in \mathcal{U}(A)$ such that

$$\|E_A(x_i a x_j) - E_A(E_P(x_i) a E_P(x_j))\|_2 < \varepsilon$$

for all $i, j = 1, \dots, k$.

Let $\varepsilon > 0$, $n \in N$ and $x_1, \dots, x_k \in M_n$ fixed. Obviously we have that there exists $l \in \mathbb{N}$ such that $n \leq l$ and

$$\|E_{\bigvee_{s=1}^l P_s}(x_i) - E_P(x_i)\|_2 < \frac{\varepsilon}{2 \max_{i=1, \dots, k} \|x_i\|^2}$$

for every $1 \leq i \leq k$.

By Theorem 2.3 we know that the triple $A_n \subseteq P_n \subseteq M_n$ verifies the relative WAHP, which implies that the triple $A_n \subseteq P_n \subseteq M_n$ also satisfies the relative WAHP, so we have that there exists $a_l \in \mathcal{U}(A_l) \subset \mathcal{U}(A)$ such that:

$$(\beta) \quad \|E_{A_l}(x_i a_l x_j) - E_{A_l}(E_{\bigvee_{s=1}^l P_s}(x_i) a_l x_j)\|_2 < \frac{\varepsilon}{2}$$

for all $i, j = 1, \dots, k$.

Proceeding like in the proof of Proposition 4.2.2. in [5] we have that the commuting square condition is preserved under inductive limit.

Consequently, we have:

$$\begin{array}{ccc} M & \supset & A \\ \cup & & \cup \\ M_l & \supset & A_l \end{array}$$

is a commuting square for all $l \in \mathbb{N}$.

The commuting square condition enable us to observe that the inequalities (β) are equivalent to

$$\|E_A(x_i a_l x_j) - E_A(E_{\bigvee_{s=1}^l P_s}(x_i) a_l x_j)\|_2 < \frac{\varepsilon}{2}$$

for all $i, j = 1, \dots, k$.

Finally, we evaluate:

$$\|E_A(x_i a_l x_j) - E_A(E_P(x_i) a_l E_P(x_j))\|_2 = \|E_A(x_i a_l x_j) - E_A(E_P(x_i) a_l x_j)\|_2 \leq$$

$$\begin{aligned} & \|E_A(x_i a_l x_j) - E_A(E_{\bigvee_{s=1}^l P_s}(x_i) a_l x_j)\|_2 + \|E_A(E_{\bigvee_{s=1}^l P_s}(x_i) a_l x_j) - E_A(E_P(x_i) a_l x_j)\|_2 < \\ & \frac{\varepsilon}{2} + \max_{i=1, \dots, k} \|x_i\|^2 \frac{\varepsilon}{2 \max_{i=1, \dots, k} \|x_i\|^2} = \varepsilon \end{aligned}$$

and we are done. □

Corollary 3.8. *Let I be a countable set, $\{M_i\}_{i \in I}$ a collection of II_1 factors and $\{A_i\}_{i \in I}$ a collection of abelian diffuse von Neumann algebras such that for every $i \in I$ we have that $A_i \subset M_i$ is a MASA. Then,*

$$\overline{\bigotimes_{i \in I} (\mathcal{N}_{M_i}(A_i))''} = (\mathcal{N}_{\overline{\bigotimes_{i \in I} M_i}}(\overline{\bigotimes_{i \in I} A_i}))''.$$

Proof. It follows immediately by applying Proposition 3.7 and Corollary 3.6 together with the fact that

$$\begin{array}{ccc} \overline{\bigotimes_{i \in S_{n+1}} M_i} & \supset & \overline{\bigotimes_{i \in S_{n+1}} A_i} \\ \cup & & \cup \\ \overline{\bigotimes_{i \in S_n} M_i} & \supset & \overline{\bigotimes_{i \in S_n} A_i} \end{array}$$

is a commuting square for all $|S_n| < \infty, S_n \subset S_{n+1} \subset I$. □

Remark 3.9.

- (1) We would like to mention that above Corollary 3.5 recovers Corollary 2.4 in [19] which is the singular version.
- (2) It is worth mentioning some questions that we believe will lead to a better understanding of these phenomenas: Is it possible to replace the commuting square condition in the Proposition 3.5 by a weaker condition, such that the same conclusion follows? If yes, what kind of condition? Is it true that we can completely drop the commuting square condition in the case where M_n are hyperfinite factors for all $n \in \mathbb{N}$?
- (3) In particular, Proposition 3.5 together with Voiculescu's famous result [21] says: in $L(\mathbb{F}_n) \overline{\otimes} L(\mathbb{F}_m)$ we cannot have a Cartan subalgebra of the form $A \overline{\otimes} B$ with $A \subset L(\mathbb{F}_n)$, $B \subset L(\mathbb{F}_m)$, which was expected.

We end this section by presenting a result that estimates the normalizing algebras for certain subalgebras of III_1 factors arising from cross-product construction.

Theorem 3.10. *Let G be a discrete ICC group and H a subgroup such that the triple $L(H) \subset L(N_G(H)) \subset L(G)$ satisfies the relative WAHP. Let N be a II_1 factor and let $A \subset N$ be a MASA. Suppose there exists an outer action $\alpha : G \rightarrow \text{Aut}(N)$ which satisfies that for all $g \in N_G(H)$ we have $\alpha_g(A) = A$.*

Then, for all $g \in N_G(H)$ we have $\alpha_g(\mathcal{N}_N(A)'') = \mathcal{N}_N(A)''$, and

$$A \rtimes_\alpha N_G(H) \subseteq \mathcal{N}_{N \rtimes_\alpha G}(A \rtimes_\alpha H)'' \subseteq \mathcal{N}_N(A)'' \rtimes_\alpha N_G(H).$$

In particular, if $A \subset N$ is a singular MASA, then

$$A \rtimes_\alpha N_G(H) = \mathcal{N}_{N \rtimes_\alpha G}(A \rtimes_\alpha H)''.$$

Proof. We only prove

$$(***) \quad \mathcal{N}_{N \rtimes_\alpha G}(A \rtimes_\alpha H)'' \subseteq \mathcal{N}_N(A)'' \rtimes_\alpha N_G(H),$$

the other parts being trivial. First let us denote by u_g the unitaries that implements the action of G on N .

In the same spirit as before, to prove $(***)$ it is enough to check that the triple

$$A \rtimes_\alpha H \subseteq \mathcal{N}_N(A)'' \rtimes_\alpha N_G(H) \subseteq N \rtimes_\alpha G$$

satisfies the relative WAHP .

Following the Remark 3.4 this is equivalent to verifying the following:

For every $S \subset G$ finite subset, every $n_g \in N$ with $g \in S$ and every $\varepsilon > 0$, there exists an

element $u \in \mathcal{U}(A \rtimes_\alpha H)$ such that

$$(\text{****}) \|E_{A \rtimes_\alpha H}(n_g u_g u n_h u_h) - E_{A \rtimes_\alpha H}(E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_g u_g) u E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_h u_h))\|_2 < \varepsilon$$

for all $g, h \in S$.

First, we fix $\varepsilon > 0$, $S \subset G$ finite subset and $n_g \in N$ with $g \in S$ and our goal is to build $u \in \mathcal{U}(A \rtimes_\alpha H)$ that will satisfies $(***)$. By the assumption that the triple $L(H) \subseteq L(N_G(H)) \subseteq L(G)$ satisfies the relative WHP, there is a unitary $v \in L(H)$ such that:

$$\|E_{L(H)}(u_g v u_h)\|_2 < \frac{\varepsilon}{2 \max_{g, h \in S} \|n_g\| \|n_h\|}$$

for g or $h \in S \setminus N_G(H)$.

If $g \in G \setminus N_G(H)$ or $h \in G \setminus N_G(H)$ we can evaluate:

$$\begin{aligned} & \|E_{A \rtimes_\alpha H}(n_g u_g a v n_h u_h) - E_{A \rtimes_\alpha H}(E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_g u_g) a v E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_h u_h))\|_2 \\ &= \|E_{A \rtimes_\alpha H}(n_g u_g a v n_h u_h)\|_2 \\ &= \|E_{A \rtimes_\alpha H}(n_g \alpha_g(a) u_g v u_h \alpha_{h^{-1}}(n_h))\|_2 \\ &\leq \|E_{N \rtimes_\alpha H}(n_g \alpha_g(a) u_g v u_h \alpha_{h^{-1}}(n_h))\|_2 \\ &= \|n_g \alpha_g(a) E_{N \rtimes_\alpha H}(u_g v u_h) \alpha_{h^{-1}}(n_h)\|_2 \\ &\leq \|n_g\| \|n_h\| \|E_{N \rtimes_\alpha H}(u_g v u_h)\|_2 \\ &= \|n_g\| \|n_h\| \|E_{L(H)}(u_g v u_h)\|_2 \\ &< \frac{\varepsilon}{2} \text{ for all } a \in \mathcal{U}(A). \end{aligned}$$

We used here that $N \rtimes_\alpha G \supset L(G)$ is a commuting square.

$$\begin{array}{ccc} \cup & & \cup \\ N \rtimes_\alpha H & \supset & L(H) \end{array}$$

So, when $g \in G \setminus N_G(H)$ or $h \in G \setminus N_G(H)$ $(***)$ holds true for any unitary of the form $u = av$ with $a \in \mathcal{U}(A)$. (δ)

When both $g, h \in N_G(H)$ we denote by $r_g = n_g - E_{\mathcal{N}_N(A)''}(n_g)$, $r_h = n_h - E_{\mathcal{N}_N(A)''}(n_h)$. Next, we approximate v by a finite sum $\sum_{k \in T} v(k) u_k$ which satisfies $\|v - \sum_{k \in T} v(k) u_k\|_2 < \frac{\varepsilon}{2}$, where T is a finite subset of H . Since $E_{\mathcal{N}_N(A)''}(r_g) = E_{\mathcal{N}_N(A)''}(r_h) = 0$ and $\mathcal{N}_N(A)'' = \alpha_{g^{-1}}(\mathcal{N}_N(A)'')$, $\mathcal{N}_N(A)'' = \alpha_k(\mathcal{N}_N(A)'')$, we obviously get:

$$E_{\mathcal{N}_N(A)''}(\alpha_{g^{-1}}(r_g)) = E_{\mathcal{N}_N(A)''}(\alpha_k(r_h)) = 0.$$

By the relative WHP for the triple $A \subseteq \mathcal{N}_N(A)'' \subseteq N$ there exists $a_\varepsilon \in \mathcal{U}(A)$ which satisfies:

$$\|E_A(\alpha_{g^{-1}}(r_g) a_\varepsilon \alpha_k(r_h))\|_2 < \frac{\varepsilon}{\#(T) \cdot \max_{k \in T} |v(k)|}.$$

At this point we can estimate:

$$\begin{aligned}
& \|E_{A \rtimes_\alpha H}(n_g u_g a_\varepsilon v n_h u_h) - E_{A \rtimes_\alpha H}(E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_g u_g) a_\varepsilon v E_{\mathcal{N}_N(A)'' \rtimes_\alpha N_G(H)}(n_h u_h))\|_2 \\
&= \|E_{A \rtimes_\alpha H}(r_g u_g a v r_h u_h)\|_2 \\
&\leq \frac{\varepsilon}{2} + \|\sum_{k \in T} v(k) E_{A \rtimes_\alpha H}(r_g u_g a_\varepsilon u_k r_h u_h)\|_2 \\
&\leq \frac{\varepsilon}{2} + \sum_{k \in T} |v(k)| \|E_{A \rtimes_\alpha H}(r_g u_g a_\varepsilon u_k r_h u_h)\|_2 \\
&= \frac{\varepsilon}{2} + \sum_{k \in T} |v(k)| \|E_{A \rtimes_\alpha H}(u_g \alpha_{g^{-1}}(r_g) a_\varepsilon \alpha_k(r_h) u_{kh})\|_2 \\
&\leq \frac{\varepsilon}{2} + \sum_{k \in T} |v(k)| \|E_{A \rtimes_\alpha N_G(H)}(u_g \alpha_{g^{-1}}(r_g) a_\varepsilon \alpha_k(r_h) u_{kh})\|_2 \\
&= \frac{\varepsilon}{2} + \sum_{k \in T} |v(k)| \|E_A(\alpha_{g^{-1}}(r_g) a_\varepsilon \alpha_k(r_h))\|_2. \quad (\rho)
\end{aligned}$$

Next, we define

- (1) $v_\varepsilon = v$ if $S \not\subseteq N_G(H)$
- (2) $v_\varepsilon = 1$ if $S \subset N_G(H)$ in which case $T = \{e\}$ and $v(e) = 1$ in the computation above.

Finally, the computation (ρ) together with (δ) show that $(***)$ holds true for $u = a_\varepsilon v_\varepsilon \in \mathcal{U}(A \rtimes_\alpha H)$, which lead to the desired conclusion. \square

To this end, for a better understanding of examples of triples of algebras that satisfy the relative WHP we would like to mention a group version for it:

Proposition 3.11. *Let $F \leq H \leq G$ groups that satisfy the following:*

(P) *For every $S \in G \setminus H$ finite subset there exists $f \in F$ such that $gfh \notin F$ for all $g, h \in S$ then the triple*

$$L(F) \subseteq L(H) \subseteq L(G)$$

satisfies the relative WHP. If in addition $F \trianglelefteq H$ is normal, then $\mathcal{N}_{L(G)}(L(F))'' = L(H)$.

Remark 3.12.

- (1) We believe that both Proposition 3.5 and Proposition 3.10 follow from a more general statement, but so far we were not able to find the right setting.
- (2) Finally, we remark that the relative WHP gives an easy path to Dixmier results [2] and also recaptures estimates of the normalizing algebra in the situations of free products in [12, 4, 6].

REFERENCES

- [1] E. Christensen, *Subalgebras of a finite algebra*, Math. Ann. **243** (1979), no. 1, 17–29. MR MR543091 (80i:46051)
- [2] J. Dixmier, *Sous-anneaux abéliens maximaux dans les facteurs de type fini*, Ann. of Math. (2) **59** (1954), 279–286. MR MR0059486 (15,539b)
- [3] H. A. Dye, *On groups of measure preserving transformations. II*, Amer. J. Math. **85** (1963), 551–576. MR MR0158048 (28 #1275)

- [4] Kenneth J. Dykema, Allan M. Sinclair, and Roger R. Smith, *Values of the Pukánszky invariant in free group factors and the hyperfinite factor*, J. Funct. Anal. **240** (2006), no. 2, 373–398. MR MR2261688
- [5] F. M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, New York, 1989. MR MR999799 (91c:46082)
- [6] A. Ioana, J. Peterson, and S. Popa, *Amalgamated Free Products of w -Rigid Factors and Calculation of their Symmetry Groups*.
- [7] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25. MR MR696688 (84d:46097)
- [8] R. V. Kadison, *Diagonalizing matrices*, Amer. J. Math. **106** (1984), no. 6, 1451–1468. MR MR765586 (86d:46056)
- [9] F. J. Murray and J. Von Neumann, *On rings of operators*, Ann. of Math. (2) **37** (1936), no. 1, 116–229. MR MR1503275
- [10] Edward Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615. MR MR0107176 (21 #5901)
- [11] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 57–106. MR MR860811 (87m:46120)
- [12] Sorin Popa, *Orthogonal pairs of $*$ -subalgebras in finite von Neumann algebras*, J. Operator Theory **9** (1983), no. 2, 253–268. MR MR703810 (84h:46077)
- [13] ———, *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 86, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995. MR MR1339767 (96d:46085)
- [14] ———, *On a class of type II_1 factors with Betti numbers invariants*, Ann. of Math. (2) **163** (2006), no. 3, 809–899. MR MR2215135 (2006k:46097)
- [15] ———, *Strong rigidity of II_1 factors arising from malleable actions of w -rigid groups. I*, Invent. Math. **165** (2006), no. 2, 369–408. MR MR2231961
- [16] ———, *Strong rigidity of II_1 factors arising from malleable actions of w -rigid groups. II*, Invent. Math. **165** (2006), no. 2, 409–451. MR MR2231962
- [17] Sorin Popa, Allan M. Sinclair, and Roger R. Smith, *Perturbations of subalgebras of type II_1 factors*, J. Funct. Anal. **213** (2004), no. 2, 346–379. MR MR2078630 (2005c:46088)
- [18] G. Robertson, Allan M. Sinclair, and Roger R. Smith, *Strong singularity for subalgebras of finite factors*, Internat. J. Math. **14** (2003), no. 3, 235–258. MR MR1978447 (2004c:22007)
- [19] A. Sinclair, R. Smith, S. White, and A. Wiggins, *Strong Singularity of Singular Masas in II_1 Factors*.
- [20] A. M. Sinclair and R. R. Smith, *Strongly singular masas in type II_1 factors*, Geom. Funct. Anal. **12** (2002), no. 1, 199–216. MR MR1904563 (2003i:46066)
- [21] D. Voiculescu, *The analogues of entropy and of Fisher’s information measure in free probability theory. III. The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1, 172–199. MR MR1371236 (96m:46119)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242

E-mail address: ichifan@math.uiowa.edu